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ON STABLE POSITION CONTROL IN DIFFERENTIAL GAMES

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A. V. KRIAZHIMSKII
(Sverdlovsk)

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A procedure for control with a guide (1 -3], yielding a solution, stable with respect to small data disturbances, of game problems of dynamics is proposed for a class, more general in comparison with [1 -3], of conflict-controlled systems, i.e., systems requiring uniqueness of the program motions and their uniform boundedness. The terminology and notation follow [2].

1. Let the conflict-controlled system

$$x^{\cdot} = f(t, x, u, v), \quad u \in P, \quad v \in Q, \quad x(t_0) = x_0 \tag{1.1}$$

($t_0 \leqslant t \leqslant \vartheta$, x is the n-dimensional phase vector, u and v are, respectively, the controls of the first and second players, P and Q are compacta in finite-dimensional Euclidean spaces, f is a continuous function) satisfy the conditions:

a) the program motion $x(t, t_0, x_*, \eta(\cdot))$, $t_0 \leqslant t \leqslant \vartheta$, exists and is unique for any initial position $\{t_0, x_*\}$ and any program control $\eta(\cdot)$;

b) the set of all program motions from positions $\{t_0. x_*\}$, $\|x_*\| < K$ ($\|\cdot\|$ is the Euclidean norm), is uniformly bounded for any $K > 0$.

Let closed sets $M, N \subset R^{n+1}$ be given (later on M_t^ε and N_t^ε denote the ε-neighborhood of sets $\{x \mid \{t, x\} \in M\}$ and $\{x \mid \{t, x\} \in N\}$, respectively) and let the first player be faced with the problem of the encounter of system (1.1) with M inside N before the instant ϑ. The first player has the following data capacity. At each current instant t the first player measures the phase state $x(t)$ of system (1.1), but not exactly: The measurement's outcome (signal) $y(t)$ is related to $x(t)$ by $\|x(t) - y(t)\| \leqslant \gamma$. In addition, in parallel with the motion of system (1.1) the first player works out and exactly measures the motion of the control system, viz., the guide

$$w^{\cdot} = f(t, w, u^*, v^*) \tag{1.2}$$

$$z^{\cdot} = f(t, y, u^*, v^*) \tag{1.3}$$

$$u^* \in P, \quad v^* \in Q, \quad y \in R^n$$

The choice of the initial state of system (1.2) -(1.3) also is at the first player's disposal. The first player's control procedure for system (1.1) within the framework of this data capacity is called the (first player's) procedure of control with a guide.

Let us consider the case when mixed controls, viz., probability (Borel, regular) measures μ on P, are admissible as the first player's controls. The control procedure S^* corresponding to this case is called a procedure of control with a guide in mixed controls and is specified by the set

$$p^* = (W^*, \mu^\circ(du \mid t, y, z), \nu^*(dv \mid t, y, z), \mu_t^*(du \mid w_*, t_*, t^*\nu)) \tag{1.4}$$

in brief notation $S^* \div p^*$. Here W^* is a closed $\tilde{u}$ -stable (relative to M) bridge contained in N and terminating on M by the instant ϑ; $\mu^\circ(du \mid t, y, z)$ and $\nu^*(dv \mid t, y, z)$ are, respectively, the mixed controls of the first and second players, such that

$$\max_{\nu\in\{\nu\}} \int_P \int_Q (y-z)' f(t, y, u, v)\, \mu^\circ(du \mid t, y, z)\, \nu(dv) =$$

$$\min_{\mu\in\{\mu\}} \max_{\nu\in\{\nu\}} \int_P \int_Q (y-z)' f(t, y, u, v)\, \mu(du)\, \nu(dv)$$

$$\min_{\mu\in\{\mu\}} \int_P \int_Q (y-z)' f(t, y, u, v)\, \mu(du)\, \nu^*(dv \mid t, y, z) =$$

$$\max_{\nu\in\{\nu\}} \min_{\mu\in\{\mu\}} \int_P \int_Q (y-z)' f(t, y, u, v)\, \mu(du)\, \nu(dv)$$

$\mu_t^*(du \mid w_*, t_*, t^*, \nu)$ is the first player's mixed control, measurable in t, with the following property: for any $\{t_*, w_*\} \in W^*$, $t^* > t_*$, and mixed control ν^* of the second player the solution $w(t)$ of the differential equation

$$w^\cdot = \int_P \int_Q f(t, w, u, v)\, \mu_t^*(du \mid w_*, t_*, t^*, \nu^*)\, \nu^*(dv)$$

$$w(t_*) = w_*$$

satisfies the condition $\{t, w(t)\} \in W^*$, $t_* \leqslant t \leqslant t^*$, if $T = \{\tau \in [t_*, t^*] \mid \{\tau, w(\tau)\} \in M\}$ is empty and the condition $\{t, w(t)\} \in W^*$, $t_* \leqslant t \leqslant \min T$ otherwise; we note that the existence of $\mu_t^*(du \mid w_*, t_*, t^*, \nu)$ follows from the properties of W^*.

A motion corresponding to the control-with-guide procedure in mixed controls $S^* \div p^*$ (1.4), to the partitioning

$$\Delta = \{t_0 = \tau_0 < \ldots < \tau_m = \vartheta\} \tag{1.5}$$

of interval $[t_0, \vartheta]$ and to data disturbance of magnitude $\gamma \geqslant 0$ is the name given to every absolutely continuous function $(x(t), w(t), z(t))$, $t_0 \leqslant t \leqslant \vartheta$, satisfying the equalities

$$x(t_0) = x_0, \quad w(t_0) = z(t_0) = w_0$$

where w_0 is the point from $W_{t_0}^* = \{w \mid \{t_0, w\} \in W^*\}$ closest to $y(\tau_0)$, $\| y(\tau_0) - x_0 \| \leqslant \gamma$, and defined on the intervals $[\tau_i, \tau_{i+1})$, $i = 0, \ldots, m-1$, by the condition: $(x(t), w(t), z(t))$, $\tau_i \leqslant t \leqslant \tau_{i+1}$, is a solution of the differential equation

$$x^\cdot = \int_P \int_Q f(t, x, u, v)\, \mu_i^\circ(du)\, \nu_t(dv)$$

$$w^\cdot = \int_P \int_Q f(t, w, u, v)\, \mu_{it}^*(du)\, \nu_i^*(dv)$$

$$z^\cdot = \int_P \int_Q f(t, y(\tau_i), u, v)\, \mu_{it}^*(du)\, \nu_i^*(dv)$$

$$\mu_i^\circ(du) = \mu^\circ(du \mid \tau_i, y(\tau_i), z(\tau_i))$$

$$\nu_i^* (dv) = \nu^* (dv \mid \tau_i, \; y(\tau_i), \; z(\tau_i))$$
$$\| y(\tau_i) - x(\tau_i) \| \leqslant \gamma$$
$$\mu_{it}^* (du) = \mu_t^* (du \mid w(\tau_i), \; \tau_i, \; \tau_{i+1}, \; \nu_i^*)$$

ν_t is some second player's mixed control measurable in t. The parameters $y(\tau_i)$, $\mu_i^\circ (du)$ and $(\mu_{it}^* (du), \nu_i^* (dv))$ mean, respectively: the signal on the state $x(\tau_i)$ of system (1.1), the mixed control constructed by the first player on the basis of data $(\tau_i, \; y(\tau_i), \; z(\tau_i))$ and applied by him to system (1.1) on the interval $[\tau_i, \; \tau_{i+1})$ and the mixed control produced by the first player on the basis of data $(\tau_i, \; \tau_{i+1}, y(\tau_i), \; w(\tau_i), \; z(\tau_i))$ and fed into system (1.3) –(1.4) (whose motion is determined further by the control $y(\tau_i)$, viz., the value of the observed signal) on the interval $[\tau_i, \tau_{i+1})$; ν_t is the realization of the second player's mixed control under the motion of system (1.1).

Theorem 1.1. Let W^* be a closed $\tilde{u}$-stable bridge contained in N and terminating on M by the instant ϑ and let $\{t_0, \; x_0\} \in W^*$. Then for any $\varepsilon > 0$ we can find $\delta > 0$ such that every motion $(x(t), \; w(t), \; z(t))$, $t_0 \leqslant t \leqslant \vartheta$, corresponding to the control-with-guide procedure in mixed controls $S^* \div p^*$ of (1.4), to partitioning (1.5) of diameter

$$d(\triangle) = \max \{\tau_{i+1} - \tau_i \mid i = 0, \; \ldots, \; m-1\} < \delta$$

and to data disturbance of magnitude $\gamma < \delta$, satisfies the condition

$$x(t) \in N_t^\varepsilon, \quad t_0 \leqslant t \leqslant \tau \leqslant \vartheta, \quad x(\tau) \in M_\tau^\varepsilon$$

As in [1 –3], the theorem's proof follows from the fact that for small $d(\Delta)$ and γ the mismatch between motion $x(t)$ and motion $w(t)$ (which by the construction of controls $\mu_{it}^* (du)$ flows along bridge W^* up to the instant of hitting onto M) is small. This fact, in its own turn, follows from the proximity, for small $d(\Delta)$ and γ, of motions $x(t)$ and $z(t)$ (ensured by the choice of controls $\mu_i^\circ (du)$ and $\nu_i^* (dv)$) on the basis of the following lemma.

Lemma 1.1. For any $\varepsilon > 0$ we can find $\delta > 0$ such that for any program controls $\eta_{(\cdot)}'$ and $\eta_{(\cdot)}''$ and any measurable n-dimensional vector-valued function $y(t)$, $t_0 \leqslant t \leqslant \vartheta$, the solution $(x(t), \; w(t), \; z(t))$, $t_0 \leqslant t \leqslant \vartheta$, of the differential equation

$$x^{\cdot} = \int_{P \times Q} f(t, x, u, v) \, \eta_t' (du, dv)$$

$$w^{\cdot} = \int_{P \times Q} f(t, w, u, v) \, \eta_t'' (du, dv)$$

$$z^{\cdot} = \int_{P \times Q} f(t, y(t), u, v) \, \eta_t'' (du, dv)$$

$$x(t_0) = x_0, \quad w(t_0) = z(t_0) = w_0$$

satisfies the condition

$$\| x(t) - w(t) \| < \varepsilon, \quad t_0 \leqslant t \leqslant \vartheta$$

if only

$$\| x_0 - w_0 \| < \delta$$

$$\| y(t) - x(t) \| < \delta; \quad \| x(t) - z(t) \| < \delta, \quad t_0 \leqslant t \leqslant \vartheta$$

(The lemma follows from properties a) and b) of system (1.1) and from the weak closedness and compactness of the set of all program controls).

N o t e 1.1. Results similar to those in[3, 4] hold for stochastic control-with-guide procedures constructed on the basis of control procedure $S^* \div p^*$ of (1.4) (see [3, 4]).

2. Now suppose that the admissible values of the first player's controls are only pure controls, i.e., elements of P. In this case, depending on whether or not the condition

$$\min_{u \in P} \max_{v \in Q} s' f(t, x, u, v) = \max_{v \in Q} \min_{u \in P} s' f(t, x, u, v), t \in [t_0, \vartheta], x, s \in R^n \quad (2.1)$$

on the saddle point in a small game is fulfilled, we use, respectively, a control procedure in pure controls and a minimax control-with-guide procedure.

We say that the program control η_t is consistent with a Borel function $v(u)$ with values in Q if we can find the first player's t-measurable mixed control $\mu_t(du)$ such that the equality

$$\int_{P \times Q} g(u, v)\, \eta_t(du, dv) = \int_P g(u, v(u))\, \mu_t(du)$$

is fulfilled for any continuous scalar function $g(u, v)$ for almost all $t \in [t_0, \vartheta]$. The control-with-guide procedure in pure controls S (the minimax control-with-guide procedure in controls S_*) is specified by the set

$$p = (W, u^\circ(t, y, z), v^*(t, y, z), \mu_t^*(du \mid w_*, t_*, t^*, v)) \quad (2.2)$$

$$(p_* = (W_*, u^\circ(t, y, z), v^*(t, y, z, u), \eta_t^*(du, dv \mid w_*, t_*, t^*, v(\cdot)))) \quad (2.3)$$

in brief notation $S \div p$ $(S_* \div p_*)$. Here W (W_*) is a closed u-stable (u_*-stable) bridge contained in N and terminating on M by the instant ϑ; $u^\circ(t, y, z) \in P$ and $v^*(t, y, z) \in Q$ $(v^*(t, y, z, u) \in Q)$ are such that

$$\max_{v \in Q} (y - z)' f(t, y, u^\circ(t, y, z), v) = \min_{u \in P} \max_{u \in Q} (y - z)' f(t, y, u, v)$$

$$\min_{u \in P} (y - z)' f(t, y, u, v^*(t, y, z)) = \max_{v \in Q} \min_{u \in P} (y - z)' f(t, y, u, v)$$

$$((y - z)' f(t, y, u, v^*(t, y, z, u)) = \max_{v \in Q} (y - z)' f(t, y, u, v)$$

where the function $v^*(t, y, z, u)$ is a Borel function in u); $\mu_t^*(du \mid w_*, t_*, t^*, v)$ is a measurable control $(\eta_t^*(du, dv \mid w_*, t_*, t^*, v(\cdot))$ is a control from the weak closure of the set of all program controls consistent with the Borel function $v(\cdot) = v(u)$ with the following property: for any $\{t_*, w_*\} \in W$ $(\{t_*, w_*\} \in W_*)$, any $t^* > t_*$ and any $v^* \in Q$ (any Borel function $v^*(u)$ with values in

Q) the solution $w(t)$ of the differential equation

$$w^{\cdot} = \int_P f(t, w, u, v^*) \mu_t^* (du \mid w_*, t_*, t^*, v^*)$$
$$(w^{\cdot} = \int_{P\times Q} f(t, w, u, v) \eta_t^* (du, dv \mid w_*, t_*, t^*, v^*(\cdot)))$$
$$w(t_*) = w_*$$

satisfies the condition $\{t, w(t)\} \in W$ $(\{t, w(t)\} \in W_*)$, $t_* \leqslant t \leqslant t^*$, if the set $T = \{\tau \in [t_*, t^*] \mid \{\tau, w(\tau)\} \in M\}$ is empty and the condition $\{t, w(t)\} \in W$ $(\{t, w(t)\} \in W_*)$, $t_* \leqslant t \leqslant \min T$, otherwise; we note that the existence of $\mu_t^* (du \mid w_*, t_*, t^*, v)$ $(\eta_t^* (du, dv \mid w_*, t_*, t^*, v(\cdot)))$ follows from the properties of W (W_*).

A motion corresponding to the control-with-guide procedure in pure strategies $S \div p$ of (2.2) (to the minimax procedure in controls $S_* \div p_*$ of (2.3)), to partitioning (1.5) of interval $[t_0, \vartheta]$ and to data disturbance of magnitude $\gamma \geqslant 0$ is the name given to every absolutely continuous function $(x(t), w(t), z(t))$, $t_0 \leqslant t \leqslant \vartheta$, which satisfies the equalities

$$x(t_0) = x_0, \quad w(t_0) = z(t_0) = w_0$$

where w_0 is the point from $W_{t_0} = \{w \mid \{t_0, w\} \in W\}$ (from $W_{*t_0} = \{w \mid \{t_0, w\} \in W_*\}$), $\| y(\tau_0) - x_0 \| \leqslant \gamma$, and which is defined on the intervals $[\tau_i, \tau_{i+1})$, $i = 0, \ldots, m-1$ by the condition: $(x(t), w(t), z(t))$, $\tau_i \leqslant t < \tau_{i+1}$ is a solution of the differential equation

$$x^{\cdot} = f(t, x, u_i^\circ, v(t))$$
$$w^{\cdot} = \int_P f(t, w, u, v_i^*) \mu_{it}^* (du)$$
$$z^{\cdot} = \int_P f(t, y(\tau_i), u, v_i^*) \mu_{it}^* (du)$$
$$u_i^\circ = u^\circ (\tau_i, y(\tau_i), z(\tau_i)), \quad v_i^* = v^* (\tau_i, y(\tau_i), z(\tau_i)),$$
$$\| y(\tau_i) - x(\tau_i) \| \leqslant \gamma$$
$$\mu_{it}^* (du) = \mu_t^* (du \mid w(\tau_i), \tau_i, \tau_{i+1}, v_i^*)$$

$v(t)$ is some second player's t -measurable control

$$(x^{\cdot} = f(t, x, u_i^\circ, v(t))$$
$$w^{\cdot} = \int_{P\times Q} f(t, w, u, v) \eta_{it}^* (du, dv)$$
$$z^{\cdot} = \int_{P\times Q} f(t, y(\tau_i), u, v) \eta_{it}^* (du, dv)$$
$$u_i^\circ = u^\circ (\tau_i, y(\tau_i), z(\tau_i)), \quad \|y(\tau_i) - x(\tau_i)\| \leqslant \gamma$$

$$\eta_{it}^* (du,\ dv) = \eta_t^* (du,\ dv \mid w (\tau_i),\ \tau_i,\ \tau_{i+1},\ v_i^* (\cdot))$$
$$v_i^* (u) = v^* (\tau_i,\ y (\tau_i),\ z (\tau_i),\ u)$$

$v\ (t)$ is some second player's t -measurable control). The following statements are valid.

T h e o r e m 2. 1. Let condition (2. 1) be fulfilled, W be a closed u-stable bridge contained in N and terminating on M by the instant ϑ and $\{t_0,\ x_0\} \in W$. Then for any $\varepsilon > 0$ we can find $\delta > 0$ such that every motion $(x\ (t),\ w\ (t),\ z\ (t))$, $t_0 \leqslant t \leqslant \vartheta$, corresponding to the control-with-guide procedure in pure controls $S \div p$ of (2.2), to partitioning (1.5) of diameter $d\ (\triangle) < \delta$ and to data disturbance of magnitude $\gamma < \delta$, satisfies the condition

$$x\ (t) \in N_t^{\varepsilon}, \quad t_0 \leqslant t \leqslant \tau \leqslant \vartheta, \quad x\ (\tau) \in M_\tau^{\varepsilon} \tag{2.4}$$

T h e o r e m 2 . 2. Let W_* be a closed u_*-stable bridge contained in N and terminating on M by the instant ϑ and $\{t_0,\ x_0\} \in W_*$. Then for any $\varepsilon > 0$ we can find $\delta > 0$ such that every motion $(x\ (t),\ w\ (t),\ z\ (t))$, $t_0 \leqslant t \leqslant \vartheta$, corresponding to the minimax control-with-guide procedure in controls $S_* \div p_*$ of (2.3), to partitioning (1.5) of diameter $d\ (\Delta) < \delta$ and to data disturbance of magnitude $\gamma < \delta$, satisfies condition (2.4).

N o t e 2. 1. Results analogous to those presented are valid for the second player's evasion problem (opposite to the encounter problem considered).

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